TOPOLOGY - III, SOLUTION SHEET 5

Exercise 1. (1) By definition, if X is contractible then there exists a point $c \in X$ such that the identity map $Id : X \to X$ is homotopy equivalent to the constant map $c : X \to X, x \mapsto c$. Then by part (1) of exercise 3, Sheet 4, it follows that $H_i(X) = H_i(c)$ for all i. In particular, $H_i(X) = 0$ for all $i > 0$.

(2) Let $q : X \times [0,1] \to CX$ be the quotient map. Then one observes that $h : CX \times [0,1] \to CX$, $(q(x,i),t) \mapsto q(x,(1-t)i)$ defines a homotopy between the identity map on CX and the constant map on CX which maps every point to the vertex of the cone.

(3) It follows from the long exact sequence of relative homology that $H_0(X,A) = 0$ if and only if the map $i_* : H_0(A) \to H_0(X)$ induced by the inclusion map $A \overset{i}{\hookrightarrow} X$ is surjective. One observes that the homomorphism $i_* : H_0(A) \to H_0(X)$ is given by the assignment $[a] \mapsto [a]$. That is i_* sends the class of a point in A to it's class in X. Recall that $H_0(Y)$ for a space Y can be interpreted as the free abelian group on all points of Y up to the identification that two points are considered equal if there is a path between them. Therefore i_* is surjective if and only if every path-connected component of X contains a point of A.

Exercise 2. (1) We have that $r_* i_*$ is the identity homomorphism on $H_n(A)$ for all n. This shows that i_* is injective, since it has a left-inverse.

(2) Let i be the usual embedding of S^1 in $\mathbb{R}^2$, then $i_* : H_1(S^1) \to H_1(\mathbb{R}^2)$ is clearly not injective since $H_1(S^1) = \mathbb{Z}$ but $H_1(\mathbb{R}^2) = 0$.

Exercise 3. (1) Let us assume that A is a collection of k points. We obtain the long exact sequence in relative homology:

$$0 \to H_2(A) \to H_2(S^2) \to H_2(S^2,A) \to H_1(A) \to H_1(S^2) \to H_1(S^2,A) \to H_0(A) \to H_0(S^2) \to H_0(S^2,A) \to 0.$$

By part (3) of exercise 1, we have that $H_0(S^2,A) = 0$. Since $H_2(A) = H_1(A) = 0$, we also obtain that $H_2(S^2,A) \cong H_2(S^2) \cong \mathbb{Z}$. Finally the vanishing of $H_1(S^2)$ gives a short exact sequence

$$0 \to H_1(S^2,A) \to H_0(A) \to H_0(S^2) \to 0.$$

Since $H_0(A) \cong \mathbb{Z}^k$ and $H_0(S^2) \cong \mathbb{Z}$, it follows that $H_1(S^2,A) \cong \mathbb{Z}^{k-1}$.

(2) Using the long exact sequence in relative homology, the fact that $\partial D^2 = S^1$ and that D^2 is contractible, it follows that $H_2(D^2, \partial D^2) = \mathbb{Z}$, $H_1(D^2, \partial D^2) = 0$ and that $H_0(D^2, \partial D^2) = 0$.

(3) Let $|A| = k$. Similar to part (1), we obtain the long exact sequence in relative homology:

$$0 \to H_2(A) \to H_2(T^2) \to H_2(T^2, A) \to H_1(A) \to H_1(T^2) \to H_1(T^2, A) \to H_0(A) \to H_0(T^2) \to H_0(T^2, A) \to 0.$$

By part (3) of exercise 1, we have that $H_0(T^2, A) = 0$. Also we have $H_2(T^2, A) \cong H_2(T^2) \cong \mathbb{Z}$. Now, the kernel of $H_0(A) \to H_0(T^2)$ is isomorphic to $\mathbb{Z}^{k-1}$. Hence we have a short exact sequence $0 \to H_1(T^2) \to H_1(T^2, A) \to \mathbb{Z}^{k-1} \to 0$. Since $H_1(T^2) \cong \mathbb{Z}^2$, we obtain that $H_1(T^2, A) \cong \mathbb{Z}^{k+1}$.

(4) First note that under the usual identifications of $H_1(S^1)$ and $H_1(T^2)$ with $\mathbb{Z}$ and $\mathbb{Z}^2$ respectively we have that i_* :Z→Z^2 is that map $c \mapsto (c, c)$. Now, we have the following long exact sequence in homology:

$$0 \to H_2(S^1) \to H_2(T^2) \to H_2(T^2, S^1) \to H_1(S^1) \xrightarrow{i_*} H_1(T^2) \to H_1(T^2, S^1) \to H_0(S^1) \to H_0(T^2) \to H_0(T^2, S^1)$$

As before we have $H_0(T^2, S^1) = 0$. Since $H_0(S^1) = \mathbb{Z}$, we also see that the map $H_0(S^1) \to H_0(T^2)$ is an isomorphism. Since i_* is injective and $H_2(T^2) \cong \mathbb{Z}$, we obtain that $H_2(T^2, S^1) \cong \mathbb{Z}$. Finally we observe that $H_2(T^2, S^1)$ is the cokernel of i_* and hence isomorphic to $\mathbb{Z}$. The same computations of $H_*(T^2, S^1)$ also hold true for the other embedding of S^1 in T^2.

(5) By 1.(3) we obtain $H_0(\mathbb{R}, \mathbb{Q}) = 0$. Note that $\mathbb{R}$ is contractible and $H_i(\mathbb{Q}) = 0$ for all $i > 0$. One can see this by using the fact that $\mathbb{Q}$ is totally disconnected. It follows at once from the long exact sequence of relative homology that $H_1(\mathbb{R}, \mathbb{Q}) \cong \mathbb{Z}^{\mathbb{Q}}$ and $H_i(\mathbb{R}, \mathbb{Q}) = 0$ for $i \neq 1$.

Exercise 4. Please refer to the proof of Theorem 2.10 on page 112 in Hatcher's book.